Therefore $L - U$ factorization would be a wasted effort. Pivoting is not necessary, because the Jacobian $[J]$ is and remains diagonally dominant. Iteration starts with all off diagonal elements set to 0. Later $[J]$ contains in its main diagonal always a sum of main diagonal elements $r_{ii}$ but off diagonal only zeros or single elements $r_{ik}$ as given in (9). Therefore simple Gauss-Jordan algorithm is superior in this application.

Due to the quadratic type of nonlinearities in defining (8) the convergence is very fast and reliable. In practical applications within line theory the root matrix is not only proven diagonally dominant but the off diagonal elements are indeed small compared with the diagonal ones. Within only two iteration steps the example from Fig. 1 shows results within $10^{-4}$ accuracy range.

## VII. Square Root of Symmetrical Matrices

The algorithm proposed in this paper applies to an arbitrary positive definite matrix. This algorithm can be simplified if in addition the matrix is symmetric, because the root matrix will also be symmetric and therefore contains fewer unknown elements. This simplification will be applicable in general transmission line theory where some calculations involve square roots of $[L]$ or $[C]$, for example $[L]^{\frac{1}{2}}[C][L]^{\frac{1}{2}}$ or $[C]^{\frac{1}{2}}[L][C]^{\frac{1}{2}}$ respectively [3]. In some special cases the product $[LC]$ is symmetric and the simplification can also be applied.

With a slight modification of the definition of the functions $f_{ik}$ to be zeroed by Newton's algorithm and a new index mapping for equations and variables it is possible to make the resulting Jacobian matrix $[J]$ also symmetrical and use specialized equation solvers to calculate the correction term $[S]$.

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# Elimination of Spurious Solutions in the Calculation of Eigenmodes by Moment Method

B. Souny, H. Aubert, and H. Baudrand

***Abstract*— In this paper the origin of nonphysical solutions obtained with Galerkin's method is described. To remove these spurious solutions a practical criterion is derived. It is shown on a patch resonator example with expansion functions satisfying edge conditions, that spurious solutions generated by the conventional approach are eliminated by the application of the proposed method.**

## I. Introduction

The problem of finding the eigenmodes of a transmission line or of a cavity is a classical one. Several papers, dealing with integral methods, describe how to determine these eigenmodes by Galerkin's method [1]–[4], but the choice of expansion functions which incorporate the edge conditions seems to generate nonphysical solutions, named spurious solutions [5].

The problem of spurious solutions has been mostly developed in the context of the finite-element method: the inaccurate approximation of the zero eigenvalue and the corresponding eigenfunctions generate spurious solutions [6]. A similar result has been obtained by the authors of the present paper in the context of transverse resonance method [7]: the inaccurate approximation of the infinite eigenvalue and the corresponding eigenfunctions generate spurious solutions.

For an integral equation the origin of spurious solution is given in [8], and a criterion for their elimination is demonstrated, but this criterion gives no practical information about the choice of expansion and weighting functions.

In this paper a practical criterion for a proper choice of the expansion functions and the weighting functions is given.

## II. Theory

### A. Notation

The extended [9] or symbolic [10] operator concept applied to moment methods allows one to use generalized expansion functions (they are in fact linear functionals), hence we further suppose here that we use an extended operator.

Extended or symbolic operator used in functional analysis is associated to the so-called transposed operator concept rather than adjoint one, as well as the use of duality product rather than scalar product [11]. Duality product is more general than scalar product; for example it is well known that it is not mathematically possible to define a scalar product involving the "Dirac function."

The notations for spaces, transposed operator and duality product are now introduced:

Let U and V represent, respectively, the domain and the range of an operator L. The elements of U and V are functions.

Let $\tilde{U}$ represent the topological dual space of U and $\tilde{V}$ represent the topological dual space of V. The elements of $\tilde{U}$ and $\tilde{V}$ are continuous linear functionals.

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$\tilde{V}$ and $\tilde{U}$ are respectively, the domain and the range of the transposed operator of L denoted by $\tilde{L}$. To define it we introduce Dirac's notation [12] which is classically used in quantum mechanic.

According to this notation, the elements of $\tilde{V}$ or $\tilde{U}$, (functionals) are denoted by $\langle \cdots |$ and the elements of V or U (functions) are denoted by $|\cdots\rangle$. Moreover, the value of a functional $\langle \cdots |$ on a function $|\cdots\rangle$ is denoted by $\langle \cdots | \cdots \rangle$. It is important to note that the symbol $\langle \cdots | \cdots \rangle$, does not necessarily designate an inner product: it is more generally a duality product (Cf. theory of distributions [11]).

One can note

$$L(|g\rangle) \overset{\text{notation}}{=} |Lg\rangle. \tag{1}$$

And $\tilde{L}$ is defined by the following relationship

$$\langle \tilde{L} f | g \rangle \overset{\text{def}}{=} \langle f | Lg \rangle \quad f \in \tilde{V}, \forall g \in U. \tag{2}$$

One can write

$$\left\{ \begin{matrix} \langle \tilde{L} f | & \overset{\text{notation}}{=} & \langle f | L \\ |Lg\rangle & \overset{\text{notation}}{=} & L|g\rangle \end{matrix} \right\} \Rightarrow \langle \tilde{L} f | g \rangle \overset{\text{def}}{=} \langle f | Lg \rangle \overset{\text{notation}}{=} \langle f | L | g \rangle. \tag{3}$$

The problem to solve consists in finding the values of the parameter $k$ which allows a nonzero solution for the following equation

$$\langle f | L(k) = 0. \tag{4}$$

Let $\{\langle H_i |\}$, $i \in [1, N]$ be a basis of elements extracted from a complete basis of $\tilde{V}$. The basis $\{\langle H_i |\}$ spans a subspace of $\tilde{V}$ denoted by $\tilde{V}_N$. Thus, an Nth order approximation $\langle f_N |$ of $\langle f |$ is given by

$$\left\langle f_N \right| = \sum_{i=1}^{N} f_i \langle H_i |. \tag{5}$$

In many cases these continuous linear functionals are defined with functions named expansion functions. However, in some cases it is not possible to associate the functional to a function, and then the continuous linear functional is often said to be a symbolic function, an example is the Dirac symbolic function. In this work the elements of $\tilde{V}_N$ are always named expansion functions.

Let $|W_j\rangle$, $j \in [1, N]$ be a basis of elements extracted from a complete basis of the space U. The basis $\{|W_j\rangle\}$ spans a subspace $U_N$ of $U$. The $|W_j\rangle$ are named weighting functions. Now, the problem (4) is replaced by the following N-order matricial equation

$$\sum_{i=1}^{N} f_i \langle H_i | L(k) | W_j \rangle = 0 \quad \forall j \in [1, N]. \tag{6}$$

### B. Origin and Elimination of Spurious Solutions

In the context of moment methods applied to the resolution of integral equations in Hilbert spaces, Schoeder and Wolf [8] have demonstrated that the occurrence of spurious solutions lies in the inaccurate choice of weighting functions for a given set of expansion functions. In this paper, a more general and practical criterion is demonstrated in normed vector spaces in which no inner product is defined (for instance, the space of distributions [11]).

Consider a basis of weighting functions $|W_j\rangle$, $j \in [1, N]$ in the subspace $U_N$. Then, there is a set of functions $\langle w_j |$, $j \in [i, N]$, composed of N functions of $\tilde{U}$ forming a reciprocal basis for a subspace $\tilde{U}_N$ of $\tilde{U}$ and given by the following relationship [10]

$$\langle w_i | W_j \rangle = \delta_{ij} \quad \forall i, j \in [1, N] \tag{7}$$

where $\delta_{ij}$ is the Kronecker delta.

Consider the operator $W$ defined by the following relationship

$$W = \sum_{i=1}^{N} |W_i\rangle \langle w_i |. \tag{8}$$

As for L, the symbol $W$ has two meanings

Applied to $|\cdots\rangle$ it is an operator fromU into $U_N$

Applied to $\langle \cdots |$ it is the transpose operator from $\tilde{U}$ into $\tilde{U}_N$

In this last case we have

$$\forall \langle T | \in \tilde{U}: \quad \left\langle T | W = \sum_{i=1}^{N} \langle T | W_i \rangle \langle w_i \right|. \tag{9}$$

Thus, the criterion given in [11] for the elimination of spurious solution can be generalized for normed vector spaces as follows

$$\{\text{range of } L \text{ on } \tilde{V}_N\} \cap \{\text{Kernel of } W \text{ on } \tilde{U}_N\} = \{0\}. \tag{10}$$

But no practical indication for the choice of weighting functions $\{|W_j\rangle\}$ is given by (10).

### C. Practical Criterion

We establish now an useful criterion for this choice from equation (10).

$\langle H_i | L$ represent the image by $L$ of $\langle H_i |$. The set $\{\langle H_i | L\}$ spans the space $I\tilde{V}_N$, range of L on $\tilde{V}_N$. One can extract from this set a subset of independent elements. The dimension of this subset is M, with M $\leq$ N (M = N if the parameter k is such that L(k) is not singular). Therefore, the elements $\{\langle H_i | L\} i \in [1, M]$ form a basis of $I\tilde{V}_N$.

If the set $\{|W_j\rangle\}$ satisfies the criterion (10) it can be demonstrated [14] that it is possible to complete the set $\{\langle H_i | L\} i \in [1, M]$ with elements of $\tilde{U}$ in order to construct a set of N linearly independent elements and generate in this way a basis of the idd subspace $\tilde{U}_N$ of $\tilde{U}$. It is useful to norm the reciprocal basis of this set and we denote it $\{|h_j\rangle\}$. The set $\{|h_j\rangle\}$ is a basis for $U_N$ (Another basis of the subspace $U_N$ is the basis $\{|W_j\rangle\}$). The transformation matrix between these two sets, is a regular matrix so the two sets are said equivalent in $U_N$.

Using the two sets $\{\langle H_i |\}$ and $\{|h_j\rangle\}$ the matrix representation of the operator is very simple. We have

$$\langle H_i | L(k) | h_j \rangle = \lambda_j \delta_{ij} \quad \lambda_j \neq 0 \quad \forall i \in [1, N], j \in [1, M] \tag{11}$$

$$\langle H_i | L(k) | h_j \rangle = 0 \quad \forall j > M \tag{12}$$

the term $\langle H_i | L(k) | i h_j' \rangle$ is the term $ij$ in the matrix representation of the operator.

So if $L(k)$ is not singular the representation is diagonal without any zero.

If $L(k)$ is singular for $i \leq M$ there are only diagonal terms $\lambda_i(k) \neq 0$ and for $i > M$ all term are null.

This demonstrate that only if $M < N$ (singular operator) the determinant of this matrix is null.

It can be demonstrated [14] that when the parameter k is close to a singular value for the operator $L$, at least one of the $\lambda_i(k)$ gets close to zero so the determinant of this matrix is continuous in the vicinity of the critical value of $k$.

So, we propose to formulate the criterion (10) as follows

Any set of weighting functions $\{|W_j\rangle\}$ must be equivalent to a set $\{|h_j\rangle\}$ defined previously.

When there is only one element in the sets $\{\langle H_i |\}$ and $\{|W_j\rangle\}$ this criterion is verified if $\langle H | L(k) | W \rangle \neq 0$ when $\langle H | L(k) \neq 0$. Therefore, no spurious solution is observed in this case.

However, the rigorous application of this criterion needs a lot of time [14] because it supposes the numerical generation of the set $\{|h_j\rangle\}$ (dependent of k).

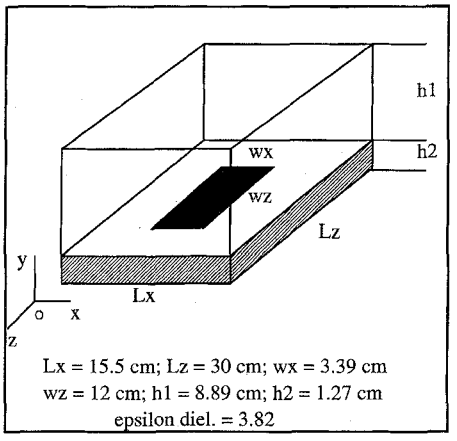


Fig. 1. Geometry of the slot resonator.

We propose subsequently to approximate $\{|h_j\rangle\}$ when $\{\langle H_i|\}$ is given (generally from physical considerations about electromagnetic fields behavior) because it can be demonstrated [14] that if the error on $\{|h_j\rangle\}$ is of the first order, the values of $\lambda_i$ are correct to the second order so that the approximate set can be chosen independent of k in a restricted range of this parameter as demonstrated in the following example.

## III. Example

The proposed method is applied to the calculation of modes in uniform wave guides such as finlines and microstrip lines, and to the determination of modes in patch and slot resonators (shielded in a metallic box). The method of resolution is the transverse resonance method [2] and [3]. For space reasons, only the results obtained in the case of patch resonator (cf. Fig. 1) are presented. This structure has been studied by Itoh [15] with one expansion function.

The expansion functions (13) have been chosen in order to satisfy edge conditions

$$H_{X,n_x,n_z} = \frac{\cos\left(\frac{n_x\pi(x-x_1)}{w_x}\right)\sin\left(\frac{n_z\pi(z-z_1)}{w_z}\right)}{\sqrt{(x_1-x)(x-x_2)}}$$

$$H_{Z,m_x,m_z} = \frac{\cos\left(\frac{m_z\pi(z-z_1)}{w_z}\right)\sin\left(\frac{m_x\pi(x-x_1)}{w_x}\right)}{\sqrt{(z_1-z)(z-z_2)}} \qquad (13)$$

$$n_x, \text{ even } \in [0,\ 2(N-1)];$$
$$m_x \text{ even } \in [2,\ 2N] m_z, \text{ odd } \in [1,\ 2N-1];$$
$$n_z \text{ odd } \in [1,\ 2N-1].$$

The determinant of the matrix generated by the moment method using these weighting functions (Galerkin's method) is given in Fig. 2.

As shown in the preceding section, for the special choice of a unique expansion function, Galerkin's method does not give spurious solutions. But for a number of expansion functions greater than one, spurious solutions appear.

The study of the image of these expansion functions by the operator L shows that it is very well approximated by the expansion functions itself in the 600–1200 MHz frequency range. So for the weighting functions we take a reciprocal set of the expansion functions set

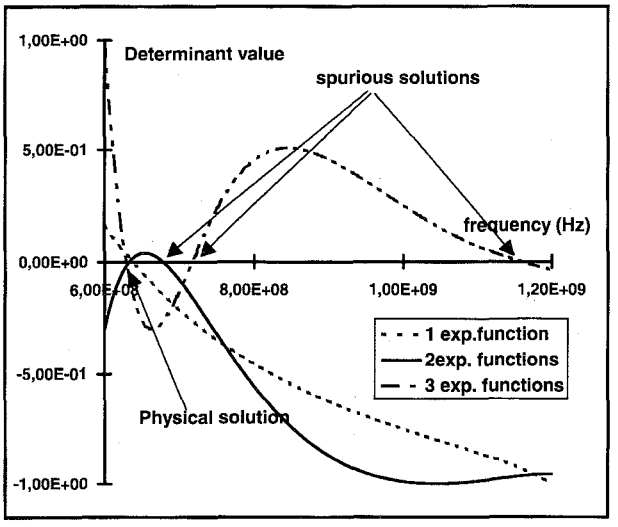


Fig. 2. Determinant value with Galerkin method.

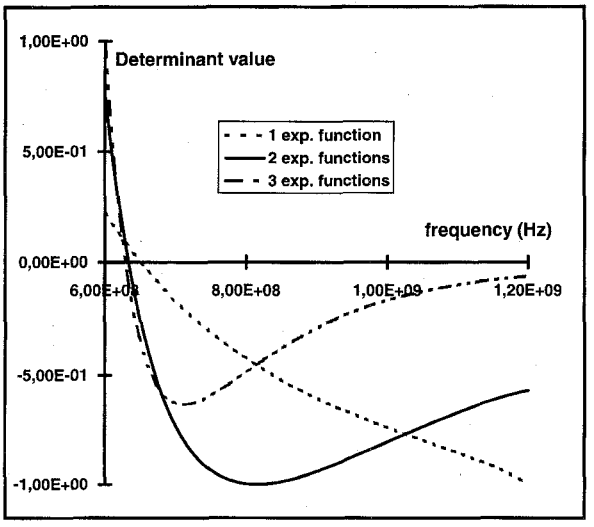


Fig. 3. Determinant value with proposed method.

(13). This set is defined by (14)

$$W_{X,n_x,n_z} = \frac{\sigma_x}{w_x}\cos\left(\frac{n_x\pi(x-x_1)}{w_x}\right)\sqrt{(x_1-x)(x-x_2)} \cdot \sin\left(\frac{n_z\pi(z-z_1)}{w_z}\right)$$

$$W_{Z,m_z,m_x} = \frac{\sigma_z}{w_z}\cos\left(\frac{m_z\pi(z-z_1)}{w_z}\right)\sqrt{(z_1-z)(z-z_2)} \cdot \sin\left(\frac{m_x\pi(x-x_1)}{w_x}\right) \qquad (14)$$

$\sigma_x$, $\sigma_z$ normalization terms

$$n_x, \text{ even } \in [0,\ 2(N-1)];$$
$$m_x \text{ even } \in [2,\ 2N] m_z, \text{ odd } \in [1,\ 2N-1];$$
$$n_z \text{ odd } \in [1,\ 2N-1].$$

The determinant of the matrix generated by the moment method using these weighting functions is given in Fig. 3. No spurious solution is detected in this case.

## IV. Conclusion

The calculation of eigenmodes in uniform wave guides or in resonant cavities by integral equation and moment method sometimes generates non physical solutions. In this paper a practical criterion for a correct choice of the weighting functions is demonstrated. Its rigorous implementation would be calculus intensive but we show that an approximate implementation efficiently eliminates spurious solutions generated by the conventional Galerkin's method without lengthening computations.

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# A New FEM Approach for Open Boundary Laplace's Problem

Dong Xingqi and An Tongyi

***Abstract*—An efficient improved finite element method (FEM) is presented for electromagnetic Laplace's problems with open boundary. The whole infinite domain is divided into a set of infinite elements instead of ordinary finite elements. Since a special FEM discretization and FEM solving procedure are used, it can not only take much less computer memory than that the conventional FEM needs, but also avoid the calculation error introduced by the truncated boundary or absorbing boundary condition used in conventional FEM.**

## I. Introduction

Recently, FEM has been more and more widely used for many electromagnetic problems with open boundary. Because in FEM the solution domain is discretized with finite elements, only finite domain problems can be handled directly. So, the solution domain with open boundary must be truncated. A relatively simple technique to implement is to select an external boundary with a zero potential to truncate the solution domain. Another alternative is to use an appropriate absorbing boundary condition or infinite element [1]–[3].

This paper presents a new FEM technique for static electromagnetic problems with open boundary. It utilizes a special discretization form to divide the whole infinite domain into infinite triangular elements. So no truncated boundary or absorbing boundary is needed and the calculation error produced by appropriate boundary condition is avoided. As an example, the capacitance matrix for the two coupled microstrips with open-boundary is calculated and the numerical results are compared with those obtained by other methods.

## II. New FEM Procedure

We begin with our discussion of a two-dimensional electrostatic open boundary problem. Assuming that the solution domain is $\Omega_s$ and its open boundary is $\Gamma_s$. We use a regular polygon $\Gamma_0$, which contains the solution domain $\Omega_s$, to divide the whole infinite domain $\Omega$ into two parts (see Fig. 1). Defining that the region within $\Gamma_0$ is $\Omega_{in}$ and the one out of $\Gamma_0$ is $\Omega_{out}$. $\Gamma_0$ may be placed very close to $\Gamma_s$. If $\Gamma_s$ itself is a regular polygon, then it is selected as $\Gamma_0$.

### A. Analysis of $\Omega_{in}$

Because the region $\Omega_{in}$ is a finite domain, it can be treated by conventional FEM. The electromagnetic field distribution can be obtained from the scalar potential $\Phi(x, y)$ satisfying the Laplace's equation with associated Dirichlet or Neumann boundary condition. The region $\Omega_{in}$ is subdivided to triangular finite elements and the stiffness matrix $K_{in}$ can be obtained by assembling each element coefficient matrix. Assuming that the number of nodes on $\Gamma_0$ is $M_0$ and the scalar potentials on $\Gamma_0$ form a column vector $\Phi_0$ of order $M_0$, the number of remaining nodes is $M_{in}$ and the corresponding potentials form a column vector $\Phi_{in}$ of order $M_{in}$. Thus, the functional within the region $\Omega_{in}$ can be obtained by

$$F_{in} = \frac{1}{2} \begin{pmatrix} \Phi_{in}^T & \Phi_0^T \end{pmatrix} \begin{pmatrix} K_{in}^{11} & K_{in}^{12} \\ K_{in}^{21} & K_{in}^{22} \end{pmatrix} \begin{pmatrix} \Phi_{in} \\ \Phi_0 \end{pmatrix}. \quad (1)$$

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